Exercises from The Rising Sea

Holonomy

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1 Chapter 2

Question 1

Let $\mathcal{O}_X$ be the pre-sheaf of smooth real-valued functions on $X = \mathbb{R}^n$. For any $p \in \mathbb{R}^n$, show that $\mathcal{O}_p$, defined as a ring of germs of smooth functions, is a local ring with maximal ideal $\mathfrak{m}_p$, the ideal of germs vanishing at p.

Answer 1

Let $\sigma \in \mathcal{O}_p$ be some germ over p, represented by a smooth function $f: U \longrightarrow \mathbb{R}$ with $U \ni p$ open, and f not vanishing at p. (Henceforth we will denote this by $\sigma = [f]_p$.)

Then since f is smooth and non-vanishing at p, there is some open nbd $V \subseteq U$ of p supporting f. Note that $[f|_V]_p = [f]_p$, by definition of germ equivalence. Thus, we can define $g \in \mathcal{O}_V$ by $g(x) = f^{-1}(x)$, and this is well defined.

Finally, consider the germ $[g]_p \in \mathcal{O}_p$. Since $g \cdot f|_V = 1$, we have $[f]_p \cdot [g]_p = 1 \in \mathcal{O}_p$.

Thus every germ over p that does not vanish at p is invertible, and $\mathfrak{m}_p$ is the unique maximal ideal as desired.

Question 2

Let $(X, \mathcal{T}\mathrm{op}_X)$ be a topological space, with $\mathcal{T}\mathrm{op}_X$ considered as the poset category of open sets of X. Show that a presheaf of sets over X is precisely a contravariant functor from $\mathcal{T}\mathrm{op}_X$ to $\mathcal{S}\mathrm{et}$

Answer 2

The data of a functor $\mathcal{F}\colon \mathcal{T}\mathrm{op}_X^{op} \longrightarrow \mathcal{S}\mathrm{et}$ is a set $\mathcal{F}(U)$ associated to each open set $U \in \mathcal{T}\mathrm{op}_X$, and a function $\mathcal{F}_{V,U}\colon \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ for each inclusion $U \subseteq V$. This is precisely the same data as a pre-sheaf $\mathcal{F}$ of sets over $(X, \mathcal{T}\mathrm{op}_X)$.

For this data to form a functor is to require the identity $U \subseteq U$ to map to the identity function, and to require the composition $U \subseteq V \subseteq W$ to map to $\mathcal{F}_{W,U} = \mathcal{F}V, U \circ \mathcal{F}W, V$. Again, these are precisely the same conditions required for $\mathcal{F}$ to form a presheaf.

Question 3

Show that the following are pre-sheaves on $\mathcal{C}$, but not sheaves.

(a) Bounded functions.

(b) Holomorphic functions admitting a holomorphic square root.

Answer 3

(a) Define $\mathcal{F}\colon \mathcal{T}\mathrm{op}_{\mathbb{C}}^{op} \longrightarrow \mathcal{S}\mathrm{et}$ by sending each complex open set $U \subseteq \mathcal{C}$ to the set of bounded functions on U. The restriction maps $\mathcal{F}_{V,U}\colon \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ are true restrictions $f \mapsto f|_U$. Function restriction respects composition, i.e. $\big(f|_V\big)|_U = f|_U$, hence $\mathcal{F}$ defined in this way is a functor, and thus a pre-sheaf.

To see that it is not a sheaf, note that an unbounded function may be bounded over every set in some open cover. (Perhaps bounded functions never form a sheaf on a non-compact space?)

For example, the norm function over $\mathcal{C}$ is bounded on restriction to any open cover of $\mathbb{C}$ by bounded opens, but is not bounded over all of $\mathcal{C}$.

(b) Now define $\mathcal{F}\colon \mathcal{T}\mathrm{op}_{\mathbb{C}}^{op} \longrightarrow \mathcal{S}\mathrm{et}$ by sending each complex open U to the set of holomorphic functions with holomorphic square root. Again, send each inclusion $U \subseteq V$ to the true restriction map. Since being holomorphic is a local property, it is preserved by restriction to open subsets, and thus this is well-defined. (Also note that the restriction of a square root will be a square root for the restriction.)

However, we may consider the identity map $z \mapsto z$. This does not have a holomorphic square root (or even a continuous square root), however, over any open set that does not contain a chosen ray from the origin, the restriction of z has a square root (a chosen branch cut). Thus, we can cover $\mathcal{C}$ with open sets where the restriction of z has a homolorphic square root, but we cannot glue those sections together. Thus, this does not form a sheaf over $\mathcal{C}$.

Question 4

Interpret the identity and gluing axioms of a sheaf $\mathcal{F}$ as saying that $\mathcal{F}(\cup_{i \in I} U_i)$ is a certain limit.

Answer 4

Consider a sheaf of sets $\mathcal{F}$ over some space X. Let U be open in X, with some open cover $\{U_i\}_{i \in I}$ for some indexing set I.

Then, consider the power set $\mathcal{P}(I)$ as a poset category by inclusion, with $J \longrightarrow J'$ when $J \subseteq J'$. Consider the diagram of sets $D_i \colon \mathcal{P}(I) \longrightarrow \mathcal{S}\mathrm{et}$ sending each subset of indices J to $\mathcal{F}\big(\bigcap_{j \in J} U_j\big)$, and each inclusion $J \longrightarrow J'$ to the restriction map.

$$\begin{tikzcd} J \arrow[d] & \mathcal{F}\big(\bigcap_{j \in J} U_j\big) \arrow[d] \\ J' & \mathcal{F}\big(\bigcap_{j \in J'} U_j\big) \end{tikzcd}$$

The bottom of this diagram then looks like

$$\begin{tikzcd} \mathcal{F}(U_i) \arrow[dr] & & \mathcal{F}(U_j) \arrow[dl] \arrow[dr] & & \mathcal{F}(U_k) \arrow[dl] \\ & \mathcal{F}(U_i \cap U_j) \arrow[dr] & & \mathcal{F}(U_j \cap U_k) \arrow[dl] & \\ & & \mathcal{F}(U_i \cap U_j \cap U_k) & & \end{tikzcd}$$

We claim that $\mathcal{F}(U)$ is the limit in $\mathcal{S}\mathrm{et}$ over this diagram.

The cone is given by the restriction maps $\mathcal{F}(U) \longrightarrow \mathcal{F}(U_J)$ for each $J \subseteq I$. By the definition of a pre-sheaf, these commute with the diagram functions, which are all also restriction maps.

To see that this is the universal cone, consider another cone $\phi_i \colon Z \longrightarrow \mathcal{F}(U_i)$.

$$\begin{tikzcd} & & Z \arrow[ddll] \arrow[dd, bend right] \arrow[ddrr] & & \\ & & \mathcal{F}(U) \arrow[dll] \arrow[d] \arrow[drr] & & \\ \mathcal{F}(U_i) \arrow[dr] & & \mathcal{F}(U_j) \arrow[dl] \arrow[dr] & & \mathcal{F}(U_k) \arrow[dl] \\ & \mathcal{F}(U_i \cap U_j) & & \mathcal{F}(U_j \cap U_k) & \end{tikzcd}$$

For each $z \in Z$, denote $z_i = \phi_i(z)$. Then since these sections agree on restriction, $z_i|_{U_i \cap U_j} = z_j|_{U_i \cap U_j}$ for each $i, j \in I$, gluing induces some section $\phi(z) \in \mathcal{F}(U)$ such that for all $i \in I$, $\phi(z)|_{U_i} = z_i$.

Define $\phi \colon Z \longrightarrow \mathcal{F}(U)$ by $z \mapsto \phi(z)$ as so. Finally, an application of identity shows that this is unique, since if $\varphi \colon Z \longrightarrow \mathcal{F}(U)$ is any other map commuting with the restriction maps, then $\varphi(z)|_{U_i} = \phi(z)|_{U_i}$ for all $i \in I$, and hence $\varphi = \phi$.

Question 5

(a) Verify that smooth functions, continuous functions, real-analytic functions, and real-valued functions on a manifold or on $\mathbb{R}^n$ form a shead.

(b) Show that real-valued continuous functions on open sets of a topological space form a sheaf.

Answer 5

... Seems boring.

Yeah, the pre-sheaves are given by the states sets. Restriction is given by true restriction. All of the properties are local in the sense that they both respect restriction, and may be determined by checking them on any cover. Real functions always glue together uniquely from their restrictions since they are determined pointwise.

Question 6

Let S be any set, and let $\mathcal{F}(U)$ be the set of maps $U \longrightarrow S$ which are locally constant. Show this is a sheaf. Equivalently, let $\mathcal{F}(U)$ be the maps $U \longrightarrow S$ which are continuous when S is endowed with the discrete topology so that every subset is continuous.

Answer 6

We take the following definition of locally constant.

$$\mathcal{F}(U) = \{f : U \longrightarrow S \mid \forall p \in S, f^{-1}(p) \in \mathcal{T}\mathrm{op}_X\}$$

Since this assigns to each open set a family of functions on it, with the restriction maps being the standard restriction maps of functions, the pre-sheaf structure and identity are both trivial.

To observe identity, consider some cover $U_i \hookrightarrow U$, then take sections $f, g : U \longrightarrow S$ such that $f_i = g_i$ for all $i \in I$ (where $f_i := f|_{U_i}$ and $g_i := g|_{U_i}$). Then for any $x \in U$, $x \in U_i$ for some i, and $f(x) = f_i(x) = g_i(x) = g(x)$.

To show gluing, take some family $f_i \in \mathcal{F}(U_i)$ such that they agree on intersections, that is,

$$\forall i, j \in I, f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

Since these are functions on U_i which agree on intersections, they glue together into a function on U.

$$f : U \longrightarrow S, \qquad x \mapsto f_i(x) \quad \text{where} \quad x \in U_i.$$

It remains to show that this is locally constant. Take any $p \in S$. Then

$$\begin{aligned} f^{-1}(p) &= \{x \in U \mid f(x) = p\} \\ &= \{x \in U \mid \forall i \in I, p \in U_i \implies f_i(x) = p\} \\ &= \bigcup_{i \in I} f_i^{-1}(p) \end{aligned}$$

Which is a union of open sets since each f_i is locally constant. Thus, f is locally constant, and the compatible f_i sections glue together.

Thus, the locally constant functions form a sheaf.

Question 7

Let X, Y be any topological spaces. Show that continuous maps to Y forms a sheaf of sets on X.

Answer 7

Again, the pre-sheaf structure is immediate because we are assigning to each U, a family of functions on U, with the standard restriction maps.

Identity is also immediate, by the same reason as the previous exercise.

Gluing is almost immediate since continuity is determined locally. Let $U_i \hookrightarrow U$ be an open cover of some open $U \subseteq X$. Then let $f_i \in \mathcal{F}(U_i)$ be continuous functions to Y agreeing on intersections in the usual way. Define $f : U \longrightarrow Y$ as in the previous question, with $f(x) = f_i(x)$ for $i \in I$ such that $x \in U_i$. Well defined since the functions agree on intersections.

Then f is continuous since for each $x \in U$, it has a neighbourhood U_i where it is continuous. Explicitly, if $V \subseteq Y$ is open, then $f^{-1}(V) = \bigcup_{i \in I} f_i^{-1}(V)$ which is open by assumption.

Question 8

Let X, Y be topological spaces.

(a) Let $\mu : Y \longrightarrow X$ be continuous. Show that section of μ form a sheaf. Explicitly, to each U open in X, assign the family of continuous maps $s : U \longrightarrow Y$ such that the following commutes.

$$\begin{tikzcd} U \arrow[r, "s"] \arrow[rr, bend right, "id|_U"'] & Y \arrow[r, "\mu"] & X \end{tikzcd}$$

(b) If Y additionally has the structure of a topological group, show that the continuous maps to Y form a sheaf of groups.

Answer 8

(a) We first confirm that restriction is well defined. Take some section $s\colon U \longrightarrow Y$, and some open subset $U' \hookrightarrow U$. Then $s|_{U'}$ is defined by precomposing s with the inclusion.

$$\begin{tikzcd} U \arrow[r, "s"] \arrow[rr, bend left, "id|_U"] & Y \arrow[r, "\mu"] & X \\ U' \arrow[u] \arrow[ur, "s|_{U'}"'] & & \end{tikzcd}$$

Thus $\mu \circ s|_{U'} = (id|_U)|_{U'} = id|_{U'}$. This shows that restriction of sections of μ is well defined, and thus sections of μ do form a pre-sheaf.

Identity is as usual trivial since we are working with functions, determined pointwise.

For gluing, take a family $s_i\colon U_i \longrightarrow Y$ which agree on intersections, and then define $s\colon U \longrightarrow Y$ to be the unique function agreeing with the s_i on restriction. Then we have

$$\begin{tikzcd} & & U_i \cap U_j \arrow[dll] \arrow[drr] & & \\ U_i \arrow[drr] \arrow[dd, "s_i"'] & & & & U_j \arrow[dll] \arrow[dd, "s_j"] \\ & & U \arrow[dll, "s"] \arrow[drr, "s"'] & & \\ Y \arrow[d, "\mu"'] & & & & Y \arrow[d, "\mu"] \\ X \arrow[rrrr, equal] & & & & X \end{tikzcd}$$

It remains to show that s is a section. Take any $x \in U$, then $x \in U_i$ for some $i \in I$, and $\mu \circ s(x) = \mu \circ s_i(x) = x$. Thus, yes, the compatible s_i glue together into a section s, and we have shown that sections of μ form a sheaf

(b) Now assume that Y is a topological group. That is, that it has a continuous multiplication operation $\times\colon Y \times Y \longrightarrow Y$, and the inverse $(-)^{-1}\colon Y \longrightarrow Y$ is also continuous. We have already shown that continuous maps to Y form a group. It remains to show that the sets of sections form a group structure, and that restriction is a group homomorphism.

To see the first, take two sections $f, g\colon U \longrightarrow Y$. Since these are continuous maps, there is an induced map $(f, g)\colon U \longrightarrow Y \times Y$. We may then post-compose this with multiplication to get $f \times g\colon U \longrightarrow Y \times Y \longrightarrow Y$.

$$\begin{tikzcd} & & Y & \\ U \arrow[r, dashed, "{(f,g)}"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & Y \times Y \arrow[ur] \arrow[dr] \arrow[rr, "\times"] & & Y \\ & & Y & \end{tikzcd}$$

Similarly, we may define an inverse function (under this operation) as

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "\bar{f}"'] & Y \arrow[r, "(-)^{-1}"] & Y \end{tikzcd}$$

Of course, these are just applying the operation and inverse point-wise, but these constructions make it clear that $f \times g$ and $\bar{f}$ are well-defined continuous maps.